PARTIAL DIFFERENTIAL EQUATIONS

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4. Nonlinear parabolic PDE and the Navier-Stokes equations

(1) Prove the Banach fixed point theorem: Let Z be a complete normed space, and assume we have a contraction mapping, that is, $\Psi : Z \to Z$ satisfying

$$\|\Psi(v) - \Psi(w)\| \le \theta \|v - w\| \qquad \forall v, w \in Z$$

for some $\theta < 1$. Then, Ψ has a fixed point $u \in Z$, that is, $\Psi(u) = u$.

(2 points)

(2) Let $u(x,t)$ be the solution of the heat equation in $\mathbb{R}^n$

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u & = & 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) & = & u_\circ(x) & \text{for} \quad t = 0. \end{array} \right. \tag{0.1}$$

given by

$$u(x,t) = 1/(4\pi t)^{n/2} \int_{\mathbb{R}^n} u_\circ(y) e^{-\frac{|x-y|^2}{4t}} dy \tag{0.2}$$

(i) Prove that, for any bounded and uniformly continuous initial condition $u_\circ$ we have

$$\|u(t) - u_\circ\|_{L^\infty(\mathbb{R}^n)} \to 0 \qquad \text{as} \quad t \downarrow 0$$

and

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \le \|u_\circ\|_{L^\infty(\mathbb{R}^n)}$$

(ii) Deduce that, for any $k \in \mathbb{N}$, if $u_\circ \in C^k(\mathbb{R}^n)$ and all its derivatives (up to order k) are bounded and uniformly continuous then

$$\|u(t) - u_\circ\|_{C^k(\mathbb{R}^n)} \to 0 \qquad \text{as} \quad t \downarrow 0$$

and

$$\|u(t)\|_{C^k(\mathbb{R}^n)} \le \|u_\circ\|_{C^k(\mathbb{R}^n)}$$

Hint: To prove part (ii), apply (i) to the derivatives of u.

(4 points)

(3) Let $u(x,t)$ be the solution of the heat equation (0.1) in $\mathbb{R}^n$, given by (0.2).

Prove that, for any $k \in \mathbb{N}$, if $u_\circ \in C^k(\mathbb{R}^n)$ and all its derivatives (up to order k) are bounded then

$$\|u(t)\|_{C^{k+1}(\mathbb{R}^n)} \le \frac{C}{t^{1/2}} \|u_\circ\|_{C^k(\mathbb{R}^n)}$$

for some constant C depending only on n and k.

(3 points)

(4) Let $u, v \in C^\infty(\overline{\Omega} \times [0,T])$ be two solutions of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u &= f(u) \quad \text{in} \quad \Omega \times (0,T] \\ u &= 0 \quad \text{on} \quad \partial\Omega \times [0,T] \end{cases}$$

with $f \in C^\infty$. Prove that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \leq e^{Ct} \|u(0) - v(0)\|_{L^2(\Omega)}$$

for some constant C.

(4 points)

(5) Let $u \in C^\infty(\overline{\Omega} \times (0,T))$ be a solution of

$$\begin{cases} \partial_t u - \Delta u &= f(u) \quad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T) \end{cases}$$

Prove that

$$\frac{d}{dt} \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 - F(u)\right) = -\int_{\mathbb{R}^n} |\partial_t u|^2$$

where $F' = f$.

<u>*Note*</u>: This means that the "energy" of u is decreasing with time.

(3 points)

(6) A stationary solution $U(x)$ of

$$\begin{cases} -\Delta U &= f(U) \quad \text{in} \quad \Omega \\ U &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

is called *asymptotically stable* if there is an $\varepsilon > 0$ such that for any $u_\circ \in C(\overline{\Omega})$ satisfying $u_\circ = 0$ on $\partial\Omega$ and

$$\|u_\circ - U\|_{L^\infty(\Omega)} < \varepsilon$$

we have that the solution $u(x,t)$ of

$$\begin{cases} \partial_t u - \Delta u &= f(u) \quad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) &= u_\circ(x) \quad \text{for} \quad t = 0 \end{cases} \tag{0.3}$$

exists for all time (that is, $T = \infty$) and

$$\lim_{t\to\infty} u(x,t) = U(x) \qquad \text{uniformly for } x \in \Omega.$$

Use the previous exercise to prove that if $U \in C^2(\overline{\Omega})$ is an asymptotically stable solution then

$$\mathcal{E}(U) \leq \mathcal{E}(U+\eta) \quad \text{for all } \eta \in C_c^\infty(\Omega) \quad \text{with} \quad \|\eta\|_{L^\infty(\Omega)} < \varepsilon,$$

where $\mathcal{E}(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - F(w)\right)$.

(3 points)

(7) Let $u \in C(\overline{\Omega} \times [0,T])$ be a solution of

$$\begin{cases} \partial_t u - \Delta u &= -u^2 \qquad \text{in} \quad \Omega \times (0,T] \\ \qquad\quad u &= 0 \qquad\quad \text{on} \quad \partial\Omega \times (0,T]. \end{cases}$$

Prove that $u(x,T) \le 1/T$, regardless of the initial data at $t = 0$.

Hint: Use the comparison principle.

(3 points)

(8) The KPP equation

$$\begin{cases} \partial_t u - \Delta u &= u(1-u) \qquad \text{in} \quad \mathbb{R}^n \times (0,T) \\ \quad u(x,0) &= u_\circ(x) \qquad\quad \text{for} \quad t = 0 \end{cases}$$

is one of the most classical reaction-diffusion PDEs, and models population dynamics.

(i) Prove that if $0 < u_\circ(x) < 1$ for all $x \in \Omega$ then $0 < u(x,t) < 1$ for all $t > 0$, $x \in \Omega$.

(ii) Prove that for any $e \in \mathbb{S}^{n-1}$ there is a travelling-wave solution of the type

$$u(x,t) = v(x \cdot e - ct), \qquad\qquad v(z) = \frac{1}{(1 + e^{-\beta z})^2}$$

for some $\beta > 0$ and some $c > 0$.

(3 points)

(9) Derive the Navier-Stokes equations from physical principles.

(3 points)

(10) Let $\vec{w} \in C^\infty(\overline{\Omega})$ be given.

(i) Prove that there exist functions $\vec{w_\circ} : \Omega \to \mathbb{R}^n$ and $q : \Omega \to \mathbb{R}$, such that

$$\vec{w} = \vec{w_\circ} + \nabla q$$

$$\operatorname{div} \vec{w_\circ} = 0 \text{ in } \Omega$$

and with $q = 0$ on $\partial\Omega$.

(ii) Prove that such representation for $\vec{w}$ is unique. Thus, we may denote

$$\vec{w_\circ} = \Pi \vec{w},$$

the Leray projection of $\vec{w}$.

(4 points)

(11) Let $\vec{u} \in C^\infty(\overline{\Omega} \times (0,T))$ be a solution of the Navier-Stokes equations

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} &= \nu \Delta u - \nabla p \qquad \text{in} \quad \Omega \times (0,T) \\ \qquad\qquad \operatorname{div} \vec{u} &= 0 \qquad\qquad\quad\; \text{in} \quad \Omega \times (0,T) \\ \qquad\qquad\qquad \vec{u} &= 0 \qquad\qquad\quad\; \text{on} \quad \partial\Omega \times (0,T) \end{cases} \tag{0.4}$$

Prove that

$$\frac{d}{dt} \int_\Omega |\vec{u}|^2 dx \le 0$$

(3 points)

(12) Let $\vec{u} \in C^\infty(\mathbb{R}^2 \times [0,T))$ be any solution of the Navier-Stokes equations (0.4) in $\Omega = \mathbb{R}^2$, and assume that both $\vec{u}$ and its derivatives converge uniformly to zero as $|x| \to \infty$.

(i) Prove that, if we denote $\vec{u} = (u_1, u_2)$, then the vorticity

$$\omega(x,t) := \operatorname{curl} \vec{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1,$$

solves the PDE

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = \nu \Delta \omega \quad \text{in} \quad \mathbb{R}^2 \times (0,T).$$

(ii) Deduce, by using the maximum principle (see Exercise 20 from Chapter 3), that

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \le \|\omega_\circ\|_{L^\infty(\mathbb{R}^2)} \qquad \text{for all } t \in (0,T),$$

where $\omega_\circ$ is the vorticity at time $t = 0$.

Note: This is the key estimate that allows to prove global existence ($T = \infty$) for the Navier-Stokes equations in 2D.

(3 points)